Atomic Decomposition of Weighted Coorbit Spaces on Manifold

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Abstract. In the present paper, using weighted Wiener amalgams on a separable Lie group, we define general coorbit spaces on a manifold and obtain their atomic decompositions under suitable conditions in the form of analysis and synthesis properties. Our results provided extensions to the corresponding results of Dahlke et al (2004).

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1. Introduction

The celebrated Feichtinger and Gröeching theory of atomic decompositions for Banach spaces of functions or distributions on locally compact groups provides a great incentive for the study of wavelet representations and Banach frames under various abstract situations. In particular, using the concept of square - integrable group representations, the introduction of fairly general coorbit spaces on locally compact groups and the theory of frames associated with them.(cf [FG 86, 89 a,89b,92]) have played a vital role in the clarification and unification of the classical theory pertaining to wavelet analysis on Euclidean spaces and Weyl-Heisenberg groups.

However, it is known that the group representations on manifolds are not necessarily square integrable. In order to cope with this situation Ali etal [AAGM 98] have used the concept of square integrability modulo quotients. In a very recent paper, Dahlke et al [D.S.T. 03] have employed this concept to define weighted coorbit spaces on a manifold and construct approximation operators to effect atomic decompositions of these spaces. They have also discussed in detail the theory of frames for these coorbit spaces under suitable conditions.

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In the present paper our aim is to define more general weighted coorbit spaces on a manifold by replacing the weighted Banach spaces with more general weighted Wiener amalgams on a separable Lie group. The main result of this paper (Theorem 4.1) provides the analysis and synthesis properties of these generalized coorbit spaces, which correspond to a well known result of Feichtinger and Gröchenig [FG 89a, Theorem 6.1]. Our results provide extensions to the corresponding results in [D.S.T 04] in the sense that we obtain the atomic decompositions of more extensive coorbit spaces.

In section 2, we present the basic definitions and concepts for use in the sequel, while in section 3 we define the weighted Wiener amalgam spaces on a separable Lie group and use them to define weighted coorbit spaces on a manifold and obtain some basic properties of these spaces.

Section 4 contains the main results of this paper (Theorem 4.1). In this section, on the lines of [D.S.T 04],we obtain the wavelet representation of the weighted coorbit spaces $M^p_w(\mathcal{N})$ defined in the preceding section via its analysis and synthesis properties. Our result corresponds to that of Feichtinger and Gröchenig [FG 89, Theorem 6.1] on a locally compact group.

The proof of theorem 4.1 is based on three lemmas, which are given in section 5. Since we are using weighted amalgams spaces, instead of weighted Lebesgue spaces, we give proof of these lemmas in a short form.

2. Basic Definitions and Concepts

Let $\mathcal{G}$ be a separable Lie group with (right) Haar measure ν, $\mathcal{H}$ is a Hilbert space and $\mathcal{L}(\mathcal{H})$ the space of unitary operators on $\mathcal{H}$.

A mapping $\pi : \mathcal{G} \to \mathcal{L}(\mathcal{H})$ is said to be a continuous representation of $\mathcal{G}$ in $\mathcal{H}$ provided the following conditions hold:

(i) $\pi(xx') = \pi(x)\,\pi(x'); \qquad \forall\, x, x' \in \mathcal{G}.$

(ii) $\pi(e) = I_d$, where e is the identity element of $\mathcal{G}$ and I_d is the identity mapping.

(iii) For all $\phi, \psi \in \mathcal{H}$, the function $x \to \langle \phi, \pi(x)\psi\rangle_{\mathcal{H}}, \quad \forall\ x \in \mathcal{G}$, is continuous.

The representation π is called square-integrable provided it is irreducible and $\exists$ a non- zero $\psi \in \mathcal{H}$ such that

$$\int_{\mathcal{G}} |\ \langle \psi, \pi(x)\psi\rangle_{\mathcal{H}}\ |^2\ d\nu(x) < \infty.$$

The function ψ is called admissible.

Throughout this paper we assume that the Hilbert space $\mathcal{H}$ is some space $L^2(\mathcal{N})$ defined on a manifold $\mathcal{N}$. As pointed out in [D.S.T 03,p.4], in a number of cases of practical importance the conditions of square integrability does not hold. In order to cope up with such problems we restrict Π to a convenient quotient group $\mathcal{G}/P$, P being a closed subgroup of $\mathcal{G}$. In the sequel we shall use right coset spaces such that $x_1 \sim x_2$ if and only if $x_1 = h \; o \; x_2$ for all x_1, x_2 in $\mathcal{G}$ and $h \in P$ Also, we shall use the canonical fiber bundle structure of $\mathcal{G}$ with projection $\Pi : \mathcal{G} \to \mathcal{G}/P$ such that $\sigma : \mathcal{G}/P \to \mathcal{G}$ is a Boreal section of this fiber bundle, that is,

$$\Pi \; o \; \sigma(h) = h; \qquad\qquad \forall \; h \in \mathcal{G}/P \; .$$

We suppose that μ is a $\mathcal{G}$ -invariant measure on $\mathcal{G}/P$. On the lines of [AAGM 98], we say that an irreducible representation Π is square-integrable mod (P, σ), if $\exists$ a non-zero function $\psi \in L^2(\mathcal{N})$ such that

$$\int_{\mathcal{G}/P} | \langle f, \Pi(\sigma(h)^{-1})\psi \rangle |^2 \, d\mu(h) < \infty, \qquad \forall f \in L^2(\mathcal{N}). \tag{2.1}$$

This implies that the operator V_ψ defined by

$$V_\psi f(h) = \langle f, \Pi(\sigma(h)^{-1})\psi \rangle \tag{2.2}$$

maps $L^2(\mathcal{N})$ into $L^2(\mathcal{G}/P)$. Also, the admissibility condition in (2.1) can be expressed in the form

$$0 < \int_{\mathcal{G}/P} | \langle f, \Pi(\sigma(h)^{-1}\psi) \rangle |^2 \, d\mu(h) = \langle f, A_\sigma f \rangle < \infty, \qquad \forall f \in L^2(\mathcal{N}),$$

where A_σ is a positive, bounded and invertible operator on $L^2(\mathcal{N})$. In case $A_\sigma = \lambda I$ for some $\lambda > 0$, the Π is called strictly square -integrable mod (P$, \sigma$) and (ψ, σ) a strictly admissible pair.

Also,we assume that ψ is normalized, so that$\lambda = 1$, which ensures that the map

$$V_\psi : L^2(\mathcal{N}) \to L^2(\mathcal{G}/P) \;\text{ is an isometry.}$$

3. Weighted Coorbit Spaces on $\mathcal{G}/P$

Throughout this paper we assume that Π is a strictly square integrable representation of $\mathcal{G}$ mod (P, σ) with a strictly admissible functions ψ. Let w be a continuous submultiplicative weight function on $\mathcal{G}$, which is bounded below,i.e.,

$$w(xy) \leq w(x)w(y); \qquad \forall \; x, y \in \mathcal{G}$$

and

$$\inf_{x \in \mathcal{G}} \; w(x) \geq c_w > 0.$$

We denote by $L^p_w(\mathcal{G}/P), 1 \leq p < \infty$, the Banach space of all measurable functions f on $\mathcal{G}/P$ under the norm

$$\| f \|_{L^p_w} \equiv \| f \mid L^p_w(\mathcal{G}/P) \| = (\int_{\mathcal{G}/P} | f(h) |^p \; w^p(\sigma(h)) d\mu(h))^{1/p} < \infty. \tag{3.1}$$

The conjugate space of $L^p_w(\mathcal{G}/P)$ is the space $L^{p'}_{1/w}(\mathcal{G}/P)$, where $1/p+1/p'=1$. In case $p=\infty$, $L^\infty_w(\mathcal{G}/P)$ is the Banach space of all measurable functions f on $\mathcal{G}/P$ with respect to the norm

$$\| f \mid L^\infty_w(\mathcal{G}/P) \| = ess\sup\ _{h\in\mathcal{G}/P} \mid f(h) \mid w(\sigma(h)) < \infty. \tag{3.2}$$

We denote by l^p_w the discretized weighted sequence such that

$$l^p_w = \{c = (c_i)_{i\in I} : \| c \mid l^p_w \| = \| cw \mid l^p \|\} < \infty,$$

I being an index set.

Let Q be a compact set in $\mathcal{G}/P$ with non-empty interior and χ_{xQ} the characteristic function of xQ. Then the Wiener amalgam space $W(L^p, L^p_w)(\mathcal{G}/P)$ consists of all functions f such that

$$W(L^p, L^p_w)(\mathcal{G}/P) = \{f : \|\| f\chi_{xQ} \|_{L^p}\|_{L^p_w} < \infty\}.$$

It is well known that $W(L^p, L^p_w)$ is a Banach space and its definition is independent of the choice of Q. Also, it can be easily seen that

$$L^p_w \subseteq W(L^p, L^p_w).$$

On the lines of [DST 04,pp.5-6], we now assume that R(h,k) is a function such that

$$R(h,k) \equiv R_\psi(h,k) = \langle \Pi(\sigma(h)^{-1}\psi), \Pi(\sigma(k)^{-1})\psi\rangle \tag{3.3}$$

$$= \ < \psi, \Pi(\sigma(h)\sigma(k)^{-1})\psi >$$

$$= V_\psi(\Pi(\sigma(h)^{-1})\psi)(k). \tag{3.4}$$

From (3.3) it is clear that

$$R(h,k) = R(k,h),$$

$R(h,\cdot) \in L^2(\mathcal{G}/P)$ for any fixed $h \in \mathcal{G}/P$ and, by Schwartz's inequality,

$$R \in L^\infty(\mathcal{G}/P \times \mathcal{G}/P).$$

Through this paper we assume that

$$V_\psi(\Pi(\sigma(h)^{-1})\psi) \in L^1_w(\mathcal{G}/P) \tag{3.5}$$

with the norm bounded independently of h.

Also, for use in the sequel, we define a non-symmetric kernel $\widetilde{R}$ such that

$$\widetilde{R}(h,k) = R(h,k)\frac{w(\sigma(h))}{w(\sigma(k))}. \tag{3.6}$$

Hence, on account of the relation (3.5) and lower boundedness of the weight function, we obtain

$$\int_{\mathcal{G}/P} \widetilde{R}(h,k)d\mu(k) \leq \frac{c}{c_w} \leq c_\psi. \tag{3.7}$$

We assume that this relation holds true when integration is carried out with respect to either h or k. Also, we suppose that

$$\sup_{h,k\in\mathcal{G}} \mid \widetilde{R}(h,k) \mid \leq c_{\psi}. \tag{3.8}$$

For a fixed ψ, we now define the space $H^1_w(\mathcal{N})$ by

$$\mathcal{H}^1_w(\mathcal{N}) = \{f \in L^2(\mathcal{N}) : V_\psi f \in L^1_w(\mathcal{G}/P)\} \tag{3.9}$$

and endow it with the norm

$$\| f \mid \mathcal{H}^1_w(\mathcal{N}) = \| Vgf \mid L^1_w(\mathcal{G}/P). \tag{3.10}$$

As usual, we denote by $\mathcal{H}^{1^\sim}_w(\mathcal{N})$ the space of all conjugate linear functionals on $\mathcal{H}^1_w(\mathcal{N})$. By virtue of these definitions, it can be easily seen that the dense continuous embeddings

$$\mathcal{H}^1_w(\mathcal{N}) \hookrightarrow L^2(\mathcal{N}) \hookrightarrow \mathcal{H}^{1^\sim}_w(\mathcal{N}) \tag{3.11}$$

hold true.

Next, since $\Pi(\sigma(h)^{-1})\psi \in \mathcal{H}^1_w(\mathcal{N}), \quad \forall \quad h \in \mathcal{G}/P$, we can generalize the operators V_ψ as a sesquilinear form on on $\mathcal{H}^{1^\sim}_w(\mathcal{N}) \times \mathcal{H}^1_w(\mathcal{N})$such that

$$V_\psi(f) = \langle f, \Pi(\sigma(h)^{-1})\psi\rangle \qquad\qquad \forall \;\; f \in \mathcal{H}^{1^\sim}_w \;\text{ and }\; g \in \mathcal{H}^1_w. \tag{3.12}$$

Also, as in [DST 04], we have

$$V_\psi : \mathcal{H}^{1^\sim}_w(\mathcal{N}) \to L^\infty_{w^{-1}}(\mathcal{G}/P).$$

and

$$\widetilde{V}_\psi : L^\infty_{w^{-1}} \to \mathcal{H}^{1^\sim}_w(\mathcal{N})$$

are bounded linear operator, where $\widetilde{V}_\psi$ is given by

$$\begin{aligned}
\langle \widetilde{V_\psi} F, g\rangle_{H^{1^\sim}_w \times \mathcal{H}^1_w} &= \langle F, V_\psi g\rangle; \;\; \forall \quad F \in L^\infty_{w^{-1}}(\mathcal{G}/P) \quad \text{and} \;\; g \in \mathcal{H}^1_w(\mathcal{N}).\\
&= \int_{\mathcal{G}/P} F(k) \;\; V_\psi g(k) d\mu(k)\\
&= \int_{\mathcal{G}/P} F(k) \langle \;\; g, \Pi(\sigma(k)^{-1})\psi\rangle d\mu(k).
\end{aligned}$$

Hence we have

$$\begin{aligned}
V_\psi \widetilde{V_\psi} F &= \langle \widetilde{V_\psi} F, \Pi(\sigma(h)^{-1})\psi\rangle_{\mathcal{H}^{1^\sim}_w \times \mathcal{H}^1_w}\\
&= \langle F, V_\psi(\Pi(\sigma(h)^{-1})\psi)\rangle_{\mathcal{H}^{1^\sim}_w \times \mathcal{H}^1_w}\\
&= \langle F, R(h,.)\rangle_{\mathcal{H}^{1^\sim}_w \times \mathcal{H}^1_w}.
\end{aligned} \tag{3.13}$$

Now, on the lines of [DST 04], we define the weighted coorbit spaces $M_w^p(\mathcal{N})$ by

$$M_w^p(\mathcal{N}) = \{f \in \mathcal{H}_w^{1^\sim}\mathcal{N} : V_\psi f \in W(L^p, L_{1/w}^p)(\mathcal{G}/P)\}. \ (1 \leq p \leq \infty) \tag{3.14}$$

and equip it with the norm

$$\| f \|_{M_w^p(\mathcal{N})} = \| V_\psi f \|_{W(L^p, L_{1/w}^p)(\mathcal{G}/p)} . \tag{3.15}$$

It can be easily verified that $M_w^p(\mathcal{N})$ is a Banach space under this norm. It may be mentioned here that due to the inclusion relation

$$L_{w^{-1}}^p(\mathcal{G}/P) \subseteq L^p(\mathcal{G}/P) \subseteq W(L^p, L_{w^{-1}}^p)(\mathcal{G}/P) \tag{3.16}$$

our space $M_w^p(\mathcal{N})$ provides an extension to the space $M_{p,w}$ in [DST 04].

Next, using the reproducing kernel R, we define subspaces $\mathcal{M}_w^p(\mathcal{G}/P)$ by

$$\mathcal{M}_w^p(\mathcal{G}/P) = \{F \in W(L^p, L_w^p) : \langle F, R(h, .)\rangle = F(h)\}, 1 \leq p \leq \infty.$$

Analogous to theorem 3.1 in [DST 04], we have the following results:

Proposition 3.1. If V_ψ is define by (3.1.2) and $\widetilde{R}$ satisfies the conditions (3.7) and (3.8), then the following results hold true:

(i) If $f \in M_w^p(\mathcal{N})$, then $\langle V_\psi f, R(h, .)\rangle = V_\psi f(h)$, i.e., $V_\psi f \in \mathcal{M}_w^p(\mathcal{G}/P), 1 \leq p \leq \infty$.

(ii) $\forall\ F \in \mathcal{M}_w^p(\mathcal{G}/P), 1 \leq p \leq \infty$, $\exists$ a unique $f \in M_w^p(\mathcal{N})$ such that $F = V_\psi f$.

Hence it entails that the spaces $M_w^p(\mathcal{N})$ and $\mathcal{M}_w^p(\mathcal{G}/P)$ are isometrically isomorphic.

Proof. Although our constructions of $M_w^P(\mathcal{N})$ and $\mathcal{M}_w^p(\mathcal{G}/P)$ are similar to those in [DST 03], The involvement of $W(L^p, L_{w^{-1}}^p)(\mathcal{G}/P)$ makes them more general. Hence it is necessary to give proof of these results. □

(i) On account of the relation (3.12), we have

$$V_\psi f(h) = \langle f, \Pi(\sigma(h)^{-1})\psi\rangle_{\mathcal{H}_w^{1\sim} \times \mathcal{H}_w^1}.$$

The adjoint V_ψ^* of V_ψ is given by

$$V_\psi^* F(s) = \int_{\mathcal{G}/P} F(h)\Pi(\sigma(h)^{-1})\psi(s) d\mu(h).$$

Also. it is known that [AAGM 98] the map V_ψ can be inverted on its image by V_ψ^* such that

$$f = V_\psi^* V_\psi f = \int_{\mathcal{G}/P} \langle f, \Pi(\sigma(h)^{-1})\psi\rangle . \Pi(\sigma(h)^{-1})\psi d\mu(h).$$

Thus we see that

$$\begin{aligned} V_\psi f(h) &= \langle f, \int_{\mathcal{G}/P} R(h,k)\Pi(\sigma(k)^{-1})\psi d\mu(k)\rangle_{\mathcal{H}_w^{1\sim}\times\mathcal{H}_w^1} \\ &= \int_{\mathcal{G}/P} \overline{R(h,k)}\langle f, \Pi(\sigma(k)^{-1})\psi\rangle_{\mathcal{H}_w^{1\sim}\times\mathcal{H}_w^1} d\mu(k). \\ &= \langle V_\psi f, R(h,.)\rangle, \text{by changing the order of integration.} \end{aligned}$$

which proves (i).

(ii) We suppose that $F \in \mathcal{M}_w^p(\mathcal{G}/P), 1 \le p \le \infty$. Then we have

$$\begin{aligned} \| F \|_{L^\infty_{1/w}(\mathcal{G}/P)} &= \| \int_{\mathcal{G}/P} F(k)\overline{R(h,k)}d\mu(k) \|_{L^\infty_{1/w}(\mathcal{G}/P)} \\ &= ess \sup_{h\in\mathcal{G}/P} | \int_{\mathcal{G}/P} F(k)\overline{R(h,k)}d\mu(k) | \, w^{-1}(\sigma(h)) \end{aligned}$$

But, using Hölders inequality, we see that

$$\begin{aligned} | \int_{\mathcal{G}/P} F(k)\overline{R(h,k)}d\mu(k) | &\le \int_{\mathcal{G}/P} | F(k)\frac{1}{w(\sigma(k))} | \, R(h,k) \, | \, w(\sigma(k))d\mu(k). \\ &\le \| F \mid W(L^p, L^p_{w^{-1}}) \| \| \widetilde{R}(h,k) \mid W(L^{p'}, L^{p'}_w) \\ &\le c_\psi \| F \mid W(L^p, L^p_{w^{-1}}) \| \end{aligned}$$

Thus we obtain

$$\| F \|_{L^\infty_{1/w}(\mathcal{G}/P)} \le c_\psi \| F|W(L^p, L^p_{w^{-1}})(\mathcal{G}/P)\|.$$

$\Rightarrow F \in L^\infty_{1/w}(\mathcal{G}/P)$.

Also, from the definition of $\mathcal{M}_w^p(\mathcal{G}/P)$ and (3.13), we have

$$V_\psi \widetilde{V}_\psi F = F,$$

where $\widetilde{V}_\psi F \in \mathcal{H}_w^1(\mathcal{N})$ and $F \in W(L^p, L^p_{1/w})(\mathcal{G}/P)$.
Hence we see that $\Rightarrow \widetilde{V}_\psi F \in M_w^p(\mathcal{N})$.

The uniqueness of the function $\widetilde{V}_\psi F$ is obvious from the definition of $M_w^p(\mathcal{N})$.

4. Wavelet Representation M_w^p

In this section our aim is to find an atomic decomposition for the weighted coorbit spaces defined in the preceding section using the partition of unity on $\mathcal{G}$ as a tool.

We assume that U is a neighbourhood of the identity e in $\mathcal{G}$. A family $X = (x_i)_{i\in I}$ in $\mathcal{G}$ is called U-dense provided

$$\cup_{i\in I} x_i U = \mathcal{G}.$$

A family $X = (x_i)_{i\in I}$ in $\mathcal{G}$ is called V-separated if for some relatively compact neighbourhood V of the identity e the sets $(x_i V)_{i\in I}$ are pairwise disjoint.

The family X is called relatively separated if it is the finite union of V-separated families.

Let $C^0(\mathcal{G})$ be the linear space of complex-valued continuous functions vanishing at infinity. A family $\Phi = (\phi_i(x))_{\imath \in I}$ of functions in $C^o(\mathcal{G})$ is called a bounded uniform partition of unity of size U (U-BUPU) provided the following conditions hold:

(i) $0 \leq \phi_i(x) \leq 1; \qquad \forall\, i \in I, x \in \mathcal{G}.$

(ii) $\exists$ a U-dense and relatively separated family $X = (x_i)_{i \in I}$ in $\mathcal{G}$ such that

$$supp \;\; \phi_i(x) \subseteq x_i U, \qquad \forall\, i \in I.$$

(iii) $\sum_{i \in I} \phi_i(x) \equiv 1, \;\; \forall\, x \in \mathcal{G}.$

In fact, by virtue of the conditions (iii) we have $\cup_{i \in I} x_i U = \mathcal{G}$, which implies that the family $X = (x_i)_{i \in I}$ is U-dense.

As in [DST 04], we define the left and right U-oscillations with respect to the analyzing wavelet ψ in the form:

$$Osc_U^l(k,h) = \sup_{u \in U} \mid \langle \psi, \Pi(\sigma(k)\sigma(h)^{-1})\psi - \Pi(u^{-1}\sigma(k)\sigma(h)^{-1})\psi \rangle \mid$$

and

$$Osc_U^r(k,h) = \sup_{u \in U} \mid \langle \psi, \Pi(\sigma(k)\sigma(h)^{-1}\psi - \Pi(\sigma(k))\sigma(h)^{-1}u)\psi \rangle \mid .$$

Also, we define the w-modified U-Oscillations as:

$$osc_{U,w}^l(k,h) = osc_U^l(k,h)\frac{w(\sigma(k))}{w(\sigma(h))}$$

and

$$osc_{U,w}^r(k,h) = osc_U^r(k,h)\frac{w(\sigma(k))}{w(\sigma(h))}.$$

Further, we assume that

$$I_\sigma = \{i \in I : \sigma(\mathcal{G}/P) \cap (x_i U) \neq \phi\},$$

which insures that $x_i \in \sigma(\mathcal{G}/P)$

$$\Rightarrow \forall\, h_i \in \mathcal{G}/P \;\exists\, x_i \in \;\mathcal{G} : x_i = \sigma(h_i) \text{ for any } i \in I_\sigma.$$

$$\Rightarrow \sum_{i \in I_\sigma} \phi_i(\sigma(h)) = 1, \qquad\qquad \text{for any } h \in \mathcal{G}/p.$$

The main result of this paper is the following theorem on the wavelet representation of the space $M_w^p(\mathcal{N})$,which is analogous to Theorem 4.1 in [DST 04]:

Theorem 4.1 Let Π be a strictly square integrable representation of $\mathcal{G}$ mod (P, σ) in $L^2(\mathcal{N})$ with strictly admissible function ψ such that

$$\int_{\mathcal{G}/p} Osc_{U,w}^l(k,h)d\mu(h) \leq \gamma, \qquad\qquad (\gamma < 1), \tag{4.1}$$

$$\int_{\mathcal{G}/P} Osc^l_{U,w'}(h,k)d\mu(k) \leq \gamma, \tag{4.2}$$

$$\mu\{h \in \mathcal{G}/P : \sigma(h) \in \sigma(h_i)Q\} \geq C_Q > 0, \tag{4.3}$$

for all $i \in I_\sigma$, Q being any compact neighbourhood of $e \in \mathcal{G}$, and

$$\int_{\mathcal{G}/P} \sup_{q\in Q} \mid \langle \Pi(\sigma h)^{-1}\psi, \Pi(\sigma(k)^{-1}q)\psi\rangle \mid \frac{w(q^{-1}\sigma(k))}{w(\sigma(h))} d\mu(k) \leq C_Q, \tag{4.4}$$

where $C_Q, C_Q^{'}$ are constants independent of h. Then the following results hold true:

(i) Every $f \in M^p_w(\mathcal{N}), 1 \leq p \leq \infty$, has a representation

$$f = \sum_{i\in I_\sigma} c_i(f)\psi_i \tag{4.5}$$

with $(c_i(f))_{i\in I_\sigma} \in W(L^p, l^p_{1/w})$ and

$$\| (c_i(f))_{i\in I_\sigma} \mid W(L^p, l^p_{1/w}) \leq \ A \| f \|_{M^p_w}, \tag{4.6}$$

where

$$c_i(f) = \langle f, \psi_i\rangle_{\mathcal{H}^{1\sim}_w \times \mathcal{H}^1_w}, \tag{4.7}$$

$$\psi_i = \Pi(\sigma(h_i)^{-1})\psi, \ \ \forall \ i \in I_\sigma, \tag{4.8}$$

and A is a positive constant .

(ii) If Q is any compact neighbourhood of the identity in $\mathcal{G}$ and ψ satisfies the condition

$$\int_{\mathcal{G}/P} \sup_{q\in Q} \mid \langle \Pi(\sigma(h)^{-1})\psi, \Pi(\sigma(k)^{-1}q)\psi\rangle \mid \frac{w(q^{-1}\sigma(k))}{w(\sigma(h))} d\mu(k) \leq C_Q^{''}, \tag{4.9}$$

where $C_Q^{"}$ is a constant independent of h, then any f which can be expressed in the form

$$f = \sum_{i\in I_\sigma} c_i(f)\psi_i \tag{4.10}$$

is contained in $M^p_w(\mathcal{N})$ with

$$\| f \|_{M^p_w} \leq B \| (c_i(f))_{i\in I_\sigma} \mid W(L^p, l^p_{\frac{1}{w}}) \|, \tag{4.11}$$

B being a positive constant.

It may be mentioned here that due to our construction of $M^p_w(N)$ the above theorem provides an extension of the corresponding result in [DST 04, Theorem 4.1].

Supporting Lemmas. We shall use the following lemmas in the proof of our theorem.

Lemma 5.1. Assume that $(X, \mathcal{A}, \eta)$ and (Y, B, ξ) are σ finite measure spaces, K is an $\mathcal{A} \bigotimes \mathcal{B}$ -measurable function on $X \times Y$ and w a positive weight function such that the following conditions hold:

$$\int_X \mid K(x,y) \mid \frac{w(y)}{w(x)} d\eta(x) \leq c_k \tag{5.1}$$

for almost all $y \in Y$

and

$$\int_Y \mid K(x,y) \mid \frac{w(y)}{w(x)} d\xi(y) \leq c_k \tag{5.2}$$

for almost all $x \in X$.

If $f \in W(L^p, L^p_{w^{-1}})(X)$, $1 \leq p \leq \infty$, then the integral

$$Tf(x) = \int_Y K(x,y) f(y) d\xi(y)$$

is absolutely convergent for almost all $x \in X$, $Tf \in W(L^p, L^p_{1/w})(X)$ and

$$\| Tf \mid W(L^p, L^p_{1/w})(X) \| \leq c_k \| f \mid W(L^p, L^p_{1/w})(X) \| .$$

Proof. We have

$$\begin{aligned}
\| Tf \mid W(L^p, L^p_{1/w}) \|^p &= \| \| Tf \chi_{xQ} \|_{L^p} \|_{L^p_{w^{-1}}} \\
&= \int_X (\int_X \mid Tf \mid^p \chi_Q(t-x) d\eta(t)) w^{-p}(x) d\eta(x) \\
&= \int_X (\int_X \mid \int_Y K(x,y) f(y) d\xi(y) \mid^p \chi_Q(t-x) d\eta(t)) \\
&\quad \times W^{-p}(x) d\eta(x) \\
&\leq \int_X \left(\int_X (\int_Y (\mid K(x,y) \mid w(y)^{1/p + 1/p'} . \frac{\mid f(y) \mid}{w(y)} d\xi(y) \right)^p \\
&\quad \chi_Q(t-x) d\eta(t)) w^{-p}(x) d\eta(x) \\
&\leq \int_X (\int_X (\int_Y \mid K(x,y) \mid w(y) \\
&\quad \frac{\mid f(y) \mid^p}{w^p(y)} d_\xi(y))^{p/p'} X \\
&\quad (\int_Y \mid K(x,y) \mid w(y) d\xi(y))^{p/p'} \chi_Q(t-x) d\eta(t) \\
&\quad \times \frac{1}{w^p(x)} d\eta(x). \\
&\leq C_k^{p/p'} \int_X \mid (\int_X \mid f(t) \mid^p \chi_Q(t-x) d\eta(t)) w^{-p}(x) d\eta(x) \\
&= C_K^{p/p'} \| f \mid W(L^p, L^p_{w^{-1}})(X) \|^p .
\end{aligned}$$

□

Lemma 5.2. Let $T_\phi : \mathcal{M}_w^p(\mathcal{G}/P) \to \mathcal{M}_w^p(\mathcal{G}/P)$ be any operator such that

$$\begin{aligned} T_\phi F(h) &= \sum_{i\in I_\sigma} \langle F, \phi_i o\sigma\rangle R(h_i, h) \\ &= \sum_{i\in I_\sigma} \int_{\mathcal{G}/P} F(k)\phi_i(\sigma(k))d\mu(k)R(h_i, h) \end{aligned}$$

$\sum_{i\in I_\sigma}$ being the limit of the partial sums over finite subsets of I_σ and ϕ an arbitrary U-BUPU, and the conditions (4.1) and (4.2) hold true, then the operator T_ϕ is bounded with bounded inverse.

Proof. Let $f \in \mathcal{M}_w^p(\mathcal{G}/P)$. Then as in [DST 03,p.14], we have

$$\begin{aligned} \mid F(h) - T_\phi F(h) \mid &= \sum_{i\in I_\sigma} \int_{\mathcal{G}/P} F(k)\phi_i(\sigma(k))[R(k,h) - R(h_i, h)d\mu] \\ &\leq \sum_{i\in I_\sigma} \int_{\mathcal{G}/P} \mid F(k) \mid \phi_i(\sigma(k)) \mid R(k,h) - R(h_i,h) \mid d\mu \\ &= \sum_{i\in I_\sigma} \mid F(k) \mid \phi_i(\sigma(k)). \mid \langle \psi, \Pi(\sigma(k)\sigma(h)^{-1})\psi \\ &\quad -\Pi(\sigma(h_i)\sigma(h)^{-1})\psi\rangle \mid d\mu(k) \\ &\leq \sum_{i\in I_\sigma} \int_{\mathcal{G}/P} \mid F(k)\phi_i(\sigma(k))Osc_u^l(k,h)d\mu(k), \\ &\quad \text{for } \ \sigma(h_i) = u^{-1}\sigma(k). \\ &= \int_{\mathcal{G}/P} \mid F(k) \mid Osc_u^l(k,h)d\mu(k). \end{aligned}$$

Thus using the hypothesis of Theorem 4.1 and Lemma 5.1, we get

$$\begin{aligned} \parallel F - T_\phi F \mid W(L^p, L^p_{1/w})(\mathcal{G}/P) \parallel &= \parallel (I - T_\phi)F \mid W(L^p, L^p_{1/w})(\mathcal{G}/P) \parallel . \\ &\leq \gamma \parallel F \mid W(L^p, L^p_{1/w})(\mathcal{G}/P) \parallel . \\ \Rightarrow \qquad\qquad \||| I - T_\phi ||| &< 1. \end{aligned}$$

$\Rightarrow I - T_\phi$ is a contraction on $\mathcal{M}_w^p(\mathcal{G}/P)$.

$\Rightarrow ||| T_\phi ||| \leq ||| T_\phi - I ||| + ||| I |||$

$\Rightarrow T_\phi$ is a bounded operator with bounded inverse.
Thus the lemma holds true. □

Lemma 5.3 If T is a bounded linear operator from $W(L^1, L^1_w)(\mathcal{G})$ into $W(L^1, l^1_w)$ with norm M_1 and from $W(L^\infty, L^\infty_w)(\mathcal{G})$ into $W(L^\infty, l^\infty_w)$ with norm M_2, then the operator T is bounded from $W(L^p, L^p_w)(\mathcal{G})$ into $W(L^p, l^p_w)$ with norm $M_1^{1/p} M_i^{(1-1/p)}$ for all $p \in \,]1, \infty[$.

Proof. The proof follows As in [DST 04, Theorem 5.3]. Since in this case weighted wiener amalgams are involved, we give a short proof. We suppose that (A_0, A_1) is an interpolation couple such that

$$A_0 = W(L^1, L^1_w)(\mathcal{G})$$

and

$$A_1 = W(L^\infty, L^\infty_w)(\mathcal{G}).$$

We write $A = A_0 + A_1$, which is a complex Banach space under the norm

$$\| a \|_A = \| a \|_{A_0 + A_1} = \inf_{a = a_0 + a_1} \{ \| a \|_{A_0}, \| a \|_{A_1} \}.$$

Let $\mathcal{S} = \{ z \in C : 0 \ Rz < 1 \}$ be a strip in the complex plane and $\mathcal{F}$ the collection of all functions f (z) defined on $\overline{\mathcal{S}}$ with values in A such that the following two conditions hold:

(i) f(z) is continuous in $\overline{\mathcal{S}}$ and analytic in $\mathcal{S}$ with

$$\sup_{z \in \overline{\mathcal{S}}} \| f(z) \|_A < \infty.$$

(ii) f(it) and f(1+it), $\forall\, t \in R$, are continuous functions in A_0 and A_1 respectively such that $\mathcal{F}$ is a Banach space under the norm

$$\| f \|_{\mathcal{F}} = \max\{ \sup_t \| f(it) \|_{A_0}, \sup_t \| f(1 + it) \|_{A_1} \} < \infty.$$

For the above interpolation couple (A_1, A_∞), we define the space $(A_0, A_1)_{[\theta]}$ by

$$(A_0, A_1)_{[\theta]} = \{ a \in A : \exists f(z) \in \mathcal{F} \text{ with } \ f(\theta) = a \}.$$

The space $(A_o, A_1)_{[\theta]}$ is a Banach space with respect to the norm

$$\| a \|_{[\theta]} = \inf\{ \| f \|_{\mathcal{F}} : f(\theta) = a \}.$$

It is known that (cf [DST 04], Theorem 5.2) if (A_0, A_1) and (B_0, B_1) are any interpolation couples and T is a linear operator from $A_o + A_1$ into $B_0 + B_1$ such that its restriction to A_j is a bounded linear operator from A_j into B_j with norm $\leq M_j, j = 0, 1$; then the restriction of T to (A_0, A_1) is a bounded linear operator from $(A_0, A_1)_{[\theta]}$ into $(B_0, B_1)_{[\theta]}$ with norm $\leq M_0^{1-\theta} M_1^{\theta}$.

Thus, by virtue of the above result, we have to prove that

$$(W(L^1, L^1_w), W(L^\infty, L^\infty_w))_{[\theta]} = W(L^p, L^p_w)(\mathcal{G}) \tag{5.3}$$

and

$$(W(L^1, l^1_w), W(L^\infty, l^\infty_w))_{[\theta]} = W(L^p, l^p_w), \tag{5.4}$$

where $\theta = 1 - 1/p$.

In order to prove (5.3), we have to show that

$$\| a \|_{[\theta]} = \| a \mid W(L^1, L^1_w)(\mathcal{G}), W(L^\infty, L^\infty_w)(\mathcal{G}) \|_{[\theta]} .$$

$$=\| a \mid W(L^p, L^p_w)(\mathcal{G}) \| . \tag{5.5}$$

Without loss of generality we may assume that

$$\| a \mid W(L^p, L^p_w)(\mathcal{G}) \| = 1.$$

We define f such that

$$f(z) = w(x)^{p(1-z)-1}(\epsilon(z^2 - \theta^2)) \mid a(x) \mid^{p(1-z)-1} a(x).$$

The function f(z) is analytic on the strip $\mathcal{S}$ and $f(\theta) = a$. Also, we see that

$$\| f \|_{\mathcal{F}} = \max\{\sup_t \| f(it) \mid W(L^1, L^1_w), \sup_t \| f(1+it) \mid W(L^\infty, L^\infty_w) \|\} \tag{5.6}$$

Now we find that

$$\begin{aligned}
\| f(it) \mid W(L^1, L^1_w) \| &= \int w(y)(\int \mid w(x)^{p(1-it)-1} \exp(\epsilon(-t^2 - \theta^2))) \\
&\quad \cdot |a(x)|^{p(1-it)-1} a(x) \chi_{yQ}(x) dx) dy. \\
&= \exp(\epsilon(-t^2 - \theta^2)) \int w^p(y)(\int \mid a(x) \mid^p \chi_{yQ}(x) dx) dy \\
&= \exp(\epsilon(-t^2 - \theta^2)) \| a \mid W(L^p, L^p_w)(\mathcal{G}) \| \\
&= \exp(\epsilon(-t^2 - \theta^2)).
\end{aligned}$$

$$\Rightarrow \sup_t \| f(it) \mid W(L^1, L^1_w) \| = \exp(-\epsilon\theta^2) \le 1. \tag{5.7}$$

Also, we have

$$\begin{aligned}
\| f(1+it) \mid W(L^\infty, L^\infty_w) \| &= \sup_y w(y)(\sup_x \mid w(x)^{p(1-(1+it))-1} \\
&\quad \exp(\epsilon(1+it)^2 - \theta^2)) \mid a(x) \mid^{p(1-(1+it))-1} \\
&\quad \times a(x) \chi_{yQ}(x)| \\
&= \exp(\epsilon(1 - t^2 - \theta^2)) \\
&\le \exp\ \epsilon.
\end{aligned} \tag{5.8}$$

On account of (5.7) and (5.8), we get

$$\| f \|_{\mathcal{F}} \le \exp\ \epsilon \to\ 1\ as\ \epsilon\ \to\ 0.$$

$$\Rightarrow \| a \|_{[\theta]} \le \| a \mid W(L^p, L^p_w) \|$$

$$\Rightarrow W(L^p, L^p_w) \subseteq (W(L^1, L^1_w), W(L^\infty, L^\infty_w))_{[\theta]}. \tag{5.9}$$

Finally, it remains to show that the converse of the inclusion in [5.9] also holds true. Without loss of generality we again assume that $\| a \|_{[\theta]} = 1$. Thus, for $1 \le p < \infty$, we have

$$\begin{aligned}
\| a \mid W(L^p, L^p_w) \| &= \sup\{| \langle a, b \rangle_w |: \| b \mid W(L^{p'}, L^{p'}_{w'}) \| = 1\}, \\
\langle a, b \rangle_w &= \int w(y)(\int a(x) b(x) w(x)^{p-1} \chi_{yQ}(x) dx) dy.
\end{aligned}$$

We write

$$F(z) = \langle f(z), g(z)\rangle_w,$$

where $f \in \mathcal{F}$ with $f(\theta) = a$,

$$g(z) = w(x)^{1-p(1-z)} \exp(\epsilon(z^2 - \theta^2)) \mid b(x) \mid^{z/(p-1)} b(x).$$

and $b \in W(L^{p'}, L^{p'}_{w'})$ with

$$\| \, b \mid W(L^{p'}, L^{p'}_{w'}) \, \|= 1.$$

Since $\| \, a \, \|_{[\theta]}= 1$, there exists $f \in \mathcal{F}$ with $f(\theta) = a$ such that

$$\| \, f(it) \mid W(L^1, L^1_w) \, \| \leq \; 1 + \epsilon$$

and

$$\| \, f(1 + it) \mid W(L^\infty, L^\infty_w) \, \| \leq \; 1 + \epsilon \qquad \text{for all } \epsilon > 0.$$

Hence we obtain

$$\begin{aligned}
\mid \mathcal{F}(it) \mid \;\; &= \;\; \mid \int w(y)(\int f(it)g(it)w(x)^{p-1}\chi_{yQ}(x)dx)dy \mid \\
&\leq \;\; \exp(\epsilon(-t^2 - \theta^2)) \\
&\quad\;\; \times \int w(y)(\int f(it)g(it)w(x)^{1-p(1-it)}w^{p-1}(x)\chi_{yQ}(x)dx)dy \\
&\leq \;\; \exp_(\, \epsilon(-t^2 - \theta^2)) \int w(y)(\int \mid f(it) \mid \chi_{yQ}(x)dx)dy
\end{aligned}$$

$f(it) \mid \chi_{yQ}(x)dx)dy$

$$\begin{aligned}
&= \;\; \exp(\epsilon(-t^2 - \theta^2)) \, \| \, f(it) \mid W(L^1, L^1_w) \, \| \\
&\leq \;\; (1 + \epsilon) \exp(-\epsilon \, \theta^2) \\
&\leq \;\; \exp \;\; \epsilon.
\end{aligned}$$

Next, we see that

$$\begin{aligned}
\mid F(1 + it) \mid \;\; &= \;\; \mid \int w(y)(\int f(1 + it)g(1 + it)w(x)^{p-1}\chi_{yQ}(x)dx)dy \mid \\
&\leq \;\; \exp(\epsilon(1 - t^2 - \theta^2)) \, \| \, f(1 + it) \mid W(L^\infty, L^\infty_w) \, \| \\
&\quad\;\; \int \mid b(x)^{p(1+it)/(p-1)}w(x)^{p(1+it)}dx \\
&\leq \;\; (1 + \epsilon) \exp(\epsilon(-t^2 - \theta^2)) \int \mid b(x)| \mid^{p/(p-1)} w(x)^p dx \\
&\leq \;\; \exp \;\; 2 \, \epsilon.
\end{aligned}$$

Hence, by (cf [DST 04]), Lemma 5.1, we have

$$\begin{aligned}
\mid F(\theta+it) \mid &\leq \exp\ 2\,\epsilon \qquad \text{for all } \theta\in[0,1] \\
&\quad \text{and } -\infty<t<\infty. \\
\Rightarrow\mid \langle a,b\rangle_w \mid &\leq \mathcal{F}(\theta)\leq \exp\ 2\,\epsilon. \\
\Rightarrow\| \ a \mid W(L^p,L^p_w)(\mathcal{G}) \ \| &\leq 1. \\
\Rightarrow (W(L^1,L^1_w),W(L^\infty,L^\infty_w))_{[\theta]} &\subseteq W(L^p,L^p_w). \qquad (5.10)
\end{aligned}$$

Combing (5.5) and (5.10), we have

$$(W(L^1,L^1_w),W(L^\infty,L^\infty_w))_{[\theta]} = W(L^p,L^p_w).$$

In the same way it can be verified that

$$(W(L^1,l^1_w),W(L^\infty,l^\infty_w))_{[\theta]} = W(L^p,l^p_w).$$

This completes the proof of the lemma. □

Proof of Theorem 4.1 (i). Using the proposition 3.1, we obtain

$$\begin{aligned}
V_\psi f(h) &= F(h) \\
&= T_\phi T_{\phi^{-1}} F(h) \\
&= \sum_{i\in I_\sigma}\langle T_\phi^{-1}F,\phi_i o\sigma\rangle R(h_i,h) \\
\Rightarrow f &= \widetilde{V_\psi}\ V_\psi\ f \\
&= \sum_{i\in I_\sigma}\langle T_\phi^{-1}F,\phi_i o\sigma\rangle \widetilde{V_\psi} R(h_i,.), \qquad (6.1)
\end{aligned}$$

because $\widetilde{V_\psi}V_\psi$ is the identity on $\mathcal{H}^{1\sim}_w(\mathcal{N})$.

Next, choosing $g\in\mathcal{H}^1_w(\mathcal{N})$, we have

$$\begin{aligned}
\langle\widetilde{V_\psi}(R(h_i,.)),g(.)\rangle_{\mathcal{H}^{1\sim}_w\times\mathcal{H}^p_w} \\
&= \langle R(h_i,.),V_\psi g(.)\rangle \\
&= \overline{V_\psi g(h_i)} \\
&= \langle\Pi(\sigma(h_i)^{-1})\psi,g\rangle_{\mathcal{H}^{1\sim}_w\times\mathcal{H}^1_w}
\end{aligned}$$

$$\Rightarrow \widetilde{V_\psi}(R(h_i,.)) = \Pi(\sigma(h_i)^{-1}\psi). \qquad (6.2)$$

Combining (6.1) and (6.2), the result in (4.5) holds true.

Thus, finally, it remains to show that the inequity in (4.6) holds. We give here a short proof involving Wiener amalgam spaces.

Since $X=(x_i)_{i\in I}$ is a relatively separated family, there exists a splitting $I=\cup_{r=1}^{r_o}I_r$ such that $x_iU\cap x_jU=\phi$ for $i\neq j$ and $i,j,\in I_r$.

$\Rightarrow I_\sigma = U_{r=1}^{r_0} I_{\sigma_r}$, where,

$$I_{\sigma_r} = \{i \in I_r : x_i U \cap \sigma(\mathcal{G}/P) \neq \phi\}$$

and $x_i = \sigma(h_i)$.
Now, let χ_{x_iU} be the characteristic function of x_iU. Then, for any sequence $(\eta_i)_{i\in I}$, we have

$$\begin{aligned}
\| \sum_{i\in I_\sigma} | \eta_i | \chi_{x_iU} \circ \sigma \quad | \quad & W(L^p, L^p_{w^{-1}})(\mathcal{G}/p) \|^p \\
= \quad & \||\| \sum_{i\in I_\sigma} (| \eta_i | \chi_{x_iU} o\sigma)\chi_{tQ} \|_{L^p}\|_{L^p_{w^{-1}}} \\
= \quad & \int_{\mathcal{G}/P} (\int_{\mathcal{G}/P} \sum_{i\in I_\sigma} (| \eta_i |^p \\
& \chi_{x_iU}(\sigma(h))\chi_{tQ} d\mu(t)) w^{-p}(\sigma(h)) d\mu(h), U \subseteq Q \\
= \quad & \int_{\mathcal{G}/P} (\int_{\mathcal{G}/P} \left(\sum_{r=1}^{r_o} \sum_{i\in I_{\sigma_r}} | \eta_i |^p \chi_{x_iU}(\sigma(h))\chi_{tQ}(t) d\mu(t) \right) \\
& \qquad \cdot w^{-1}(\sigma(h)) d\mu(h). \\
\geq \quad & \int_{\mathcal{G}/P} (\sum_{r=1}^{r_0} \int_{\mathcal{G}/p} (\sum_{i\in I_\sigma} | \eta_i |^p \chi_{x_iU}(\sigma(h)))\chi_{tQ} d\mu(t)) \\
& w^{-p}(\sigma(h)) d\mu(h). \\
\geq \quad & \left(\max_{x\in U} w(u) \right)^{-1} C_U \sum_{i\in I_\sigma} \| \eta_i \chi_{tQ} \|_p \, w^{-p}(x_i) \\
& \qquad \text{for } w(\sigma(h)) \leq w(x) w(x_i), \quad \forall \sigma(h) \in x_i U.
\end{aligned}$$

Since w is continuous and U is compact, using the hypothesis (4.3), we obtain

$$\| (\eta_i)_{i\in I_\sigma} \mid W(L^p, l^p_{1/w}) \| \leq c \, \| \sum_{i\in I_\sigma} | \eta_i | \chi_{x_iU} o\sigma \mid W(L^p, L^p_{1/w})(\mathcal{G}/P) \| . \tag{6.3}$$

Next, let $F \in W(L^p, L^p_{1/w})(\mathcal{G}/P)$. Then using (6.3), we have

$$\begin{aligned}
\| (\langle F, Q_i \circ \sigma\rangle)_{i\in I_\sigma} \mid W(L^p, l^p_{1/w}) \| \ \leq \ & \| (\langle | F |, \phi_i \circ \sigma\rangle)_{i\in I_\sigma} \mid W(L, l^p_{1/w}) \| \\
\leq c \, \| \sum_{i\in I_\sigma} \langle | F |, \phi_i o\sigma\rangle \chi_{x_{iU}} \circ \sigma \mid & W(L^p, L^p_{1/w})(\mathcal{G}/P) \| \qquad (6.4)
\end{aligned}$$

Now, fixing $h \in \mathcal{G}/P$, we see that

$$\begin{aligned}
\sum_{i\in I_\sigma} \langle | F |, \phi_i \circ \sigma\rangle \chi_{x_i,U}(\sigma(h)) \quad &= \quad \sum_{i\in I_h} \langle | F |, \phi_i, o\sigma\rangle, \\
&= \quad \sum_{i\in I_h} \langle | F |, \phi_i(\sigma(.))\rangle \\
&\leq \quad \langle | F |, \chi_{UU^{-1}}(\sigma(.)\sigma(h)^{-1})\rangle,
\end{aligned}$$

where $I_h = \{i \in I_\sigma : x_i \in \sigma(x)U^{-1}\}$. We observe that $\sigma(k)\sigma(h)^{-1} \in UU^{-1}$. Therefore there exists $u_1, u_2 \in U$ such that

$$\begin{aligned} \sigma(k)\sigma(h)^{-1} &= u_1 u_2^{-1} \\ \Rightarrow w(\sigma(k)) &= w(u_1, u_2^{-1}\sigma(h)) \\ &\leq w(u_1, u_2^{-1})w(\sigma(h)). \end{aligned}$$

Further, since UU^{-1} is compact and w continuous, we have

$$\frac{w(\sigma(k))}{w(\sigma(h))} \leq c,$$

where c being a positive constant independent of h and k. Thus we have

$$\begin{aligned} \int_{\mathcal{G}/P} \chi_{UU^1}(\sigma(k)\sigma(h)^{-1})\frac{w(\sigma(k))}{w(\sigma(h))}d\mu(k) &\leq c\int_{\mathcal{G}/P} \chi_{UU^{-1}}(\sigma(k)\sigma(h)^{-1})d\mu(k) \\ &\leq c, \qquad \forall\, h \ \in \mathcal{G}/P, \end{aligned}$$

and also

$$\int_{\mathcal{G}/P} \chi_{UU^{-1}}(\sigma(k)\sigma(h)^{-1})\frac{w(\sigma(k))}{w(\sigma(h))}d\mu(h) \leq c, \qquad \forall\, k \in \mathcal{G}/P.$$

Hence, by lemma 5.1, we obtain

$$\begin{aligned} \| (\langle F, \phi_i o\sigma\rangle)_{i\in I_\sigma} \mid W(L^p, l^p_{1/w}) \| &\leq A \| \langle | F |, \chi_{UU^{-1}}(\sigma(.)\sigma(h)^{-1})\rangle \\ &\quad \mid W(L^p, L^p_{1/w})(\mathcal{G}/P) \| \\ &\leq A \| \langle F \mid W(L^p, L^p_{1/w})(\mathcal{G}/P) \| \end{aligned}$$

$$\begin{aligned} &\Rightarrow \| (\langle T_\phi^{-1}V_\psi f, \phi_i o\sigma\rangle)_{i\in I_\sigma} \mid W(L^p, L^p_{1/w}) \| \\ &\qquad\qquad \leq A \| T_\phi^{-1}V_\psi f \mid W(L^P, L^p_{1/w}) \| \\ &\qquad\qquad \leq A \||\, T_\phi^{-1} \|||\, \| V_\psi f \\ &\qquad\qquad\quad \mid W(L^p, L^p_{1/w})(\mathcal{G}/P) \| \\ &\qquad\qquad \leq A \||\, T_\phi^{-1} \|||\, \| f \mid M^p_w(\mathcal{N}) \| \\ \Rightarrow \| (c_i(f))_{i\in I_\sigma} \mid WL^p, l^p_{1/w} \| &\leq A \| f \mid M^p_w(\mathcal{N}) \| . \end{aligned}$$

This completes the proof of Theorem 4.1 (i).

Proof of the Theorem 4.1(ii). At first we show that the operator $\tau : (c_i(f))_{i\in I} \to \sum_{i\in I_\sigma} c_i(f)R(h_i, .)$ is bounded from $W(L^1, l^1_{1/w})$ to $W(L^1, L^1_{1/w})(\mathcal{G}/P)$ and from $W(L^\infty, l^\infty_{1/w})$ to $W(L^\infty, L^\infty_{1/w})(\mathcal{G}/P)$. Then, by lemma 5.3, the operator τ becomes bounded from $W(L^p, l^p_{1/w})$ to $W(L^\infty, L^\infty_{1/w})(\mathcal{G}/P)$, for all $p \in [1, \infty]$.

We suppose that p=1. Then we have

$$\begin{aligned}
&\| \sum_{i\in I_\sigma} c_i(f)R(h_i,.) \quad | \quad W(L^1, L^1_{1/w})(\mathcal{G}/P) \| \\
&= \||\| \sum_{i\in I_\sigma} c_i(f)R(h_i,.)\phi_i(k) \|_{L^1}\|_{L^1_{1/w}}, \\
&\qquad\qquad (\phi_i)_{i\in I_\sigma} \text{ being } U-BUPU. \\
&= \| \int_{\mathcal{G}/P} \sum_{i\in I_\sigma} c_i(f)R(h_i,k)\phi_i(k)d\mu(k) \|_{L^1_{1/w}} \\
&\leq \sum_{i\in I_\sigma} \sup_{h,k\in\mathcal{G}/P} \int_{\mathcal{G}/P} [\Big(\int_{\mathcal{G}/P} | c_i(f)\cdot | \sup_{i\in I_\sigma} | R(h_i,k) | \\
&\quad \cdot\frac{w(\sigma(h_i))}{w(\sigma(k))}\Big)\cdot\phi_i(k)d\mu(k)] \left(\frac{w(\sigma(k))}{w(\sigma(h_i)}\right)\cdot\frac{1}{w(\sigma(h_i))} \\
&\leq c_\psi \sum_{i\in I_\sigma} \| c_i(f)\phi_i \|_{W(L^1,l^1_w)} w^{-1}(\sigma(h_i)) \\
&= c_\psi \| (c_i(f))_{i\in I_\sigma} | W(L^1, l^1_{1/w}) \| . \qquad (7.1)
\end{aligned}$$

Next, let $p=\infty$. Then we have

$$\begin{aligned}
&\| \sum_{i\in I_\sigma} c_i(f)R(h_i,h) \quad | \quad W(L^\infty, L^\infty_{1/w})(\mathcal{G}/P) \| \\
&= \||\| \sum_{i\in I_\sigma} c_i(f)R(h_i,h)\chi_{k,Q} \|_{L^\infty}\|_{L^\infty_{1/w}} \\
&\leq \| (c_i(f))_{i\in I_\sigma} \|_{L^\infty} \sup_{i\in I_\sigma}(w^{-1}\sigma(h_i)). \\
&\quad \sup_{h\in\mathcal{G}/P} \sum_{i\in I_\sigma} | R(h_i,h) | \frac{w(\sigma(h_i))}{w(\sigma(h))} \\
&= \| (c_i(f))_{i\in I} | W(L^\infty, l^\infty_{1/w}) \| . \\
&\quad \sup_{h\in\mathcal{G}/P} \sum_{i\in I_\sigma} | \widetilde{R}(h_i,h) | . \qquad (7.2)
\end{aligned}$$

Further, using (4.9), it can be verified that (cf[DST 03];p.20)

$$\sup_{h\in\mathcal{G}/P} \sum_{i\in I_\sigma} | \widetilde{R}(h_i,h) | \leq \frac{r_o C''_Q}{C_Q}, \qquad (7.3)$$

where $r_o \in I_\sigma$ is a fixed number.

Combing (7.2) and (7.3), we obtain

$$\| \sum_{i\in I_\sigma} c_i(f)R(h_i,h) | W(L^\infty, L^\infty_{1/w})(\mathcal{G}/P) \| \leq \| c_i(f)_{i\in I} | W(L^\infty, l^\infty_{1/w}) \| \frac{r_o c''_Q}{c_Q}. \qquad (7.4)$$

Thus, on account of Lemma 5.3, we have

$$\| \sum_{i\in I_\sigma} c_i(f)R(h_i,h) \mid W(L^p, L^p_{1/w})(\mathcal{G}/P) \| \leq B \| \sum_{i\in I_\sigma} c_i(f)_{i\in I_\sigma} \mid W(L^p, l^p_{1/w}) \| . \tag{7.5}$$

Finally, using Proposition 3.1, we get

$$\begin{aligned} \sum_{i\in I_\sigma} c_i(f)R(h_i,h) &= V_\psi \widetilde{V_\psi} \sum_{i\in I_\sigma} c_i(f) V_\psi(\Pi(\sigma(h_i)^{-1})\psi) \\ &= V_\psi(\sum_{i\in I\sigma} c_i \Pi(\sigma(h_i)^{-1})\psi). \\ \Rightarrow \| \sum_{i\in I_\sigma} c_i(f)(\Pi(\sigma(h_i)^{-1})\psi) \mid M^p_w \| &= \| \sum_{i\in I_\sigma} c_i(f)R(h_i,h) \mid W(L^p, L^p_{1/w}) \| \\ \Rightarrow \| f \|_{M^p_w} &\leq B \| (c_i(f))_{i\in I_\sigma} \mid W(L^p, l^p_{1/w}) \| . \end{aligned}$$

This completes the proof of the theorem.

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